New Insights in Loop Quantum Cosmology through an Exactly Solvable Model

(Work in collaboration with Abhay Ashtekar and Alejandro Corichi)

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Motivation

Extensive analytical and numerical methods in LQC: valuable insights on singularity resolution in symmetry reduced models.

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- Underlying discrete quantum geometry leads to repulsive QG effects at Planck scale. **Classical Big Bang** replaced by **Quantum Big Bounce**.
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  - In what sense LQC and WDW converge to each other or diverge from each other?
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  - What happens to the fluctuations in general? Does universe retain its memory through the bounce?
  - In what sense LQC and WDW converge to each other or diverge from each other?
  - Does LQC have a quantum continuum limit?
**Exactly Solvable LQC (SLQC)**

- Canonical quantization of homogeneous and isotropic cosmology based on LQG.

- Homogeneity and Isotropy $\Rightarrow A^i_a \rightarrow c$, $E^a_i \rightarrow p$

Relation with metric variables: $|p| = a^2$, $c \propto \dot{a}$

- Full control on quantum theory for various models.

Quantum constraint with massless scalar:

$$\Theta(v)\Psi(v, \phi) = -\partial^2_\phi \Psi(v, \phi)$$

Uniform difference equation in $v$ with a correct classical limit, no gauge artifacts and no fake Planck scale effects. (Contrast with early models, other discretization schemes).

$$v = |p|^{3/2} / (2\pi \gamma \ell_P^2), \quad b := c/|p|^{1/2}, \quad \{b, v\} = 2$$

- Quantum Constraint in $b$ representation:

$$\Theta(b)\chi(b, \phi) = -12\pi G \frac{\sin(\lambda b)}{\lambda} \left( \frac{\partial}{\partial b} \frac{\sin(\lambda b)}{\lambda} \frac{\partial}{\partial b} \right) \chi(b, \phi) = -\partial^2_\phi \chi(b, \phi)$$

$\phi \rightarrow$ internal time. $\Theta$: positive definite & self adjoint. $\lambda^2 \rightarrow$ Area Gap
Hilbert space can be constructed following Klein-Gordon theory (Positive frequency solutions).

Physical Inner product:

\[(\chi, \chi)_{\text{phy}} = \int d\bar{b} \bar{\chi}(b)|\hat{v}|\chi(b)\]

Dirac Observables: \(\hat{P}_\phi, \hat{V}|\phi\)

Introduce \(x := (12\pi G)^{-1/2} \ln(\tan(\lambda b/2))\)

Quantum Constraint: \(\partial^2_\phi \chi(\phi, x) = \partial^2_x \chi(\phi, x)\)

General solution:

\[\chi = \chi_+(\phi + x) + \chi_-(\phi - x) := \chi_+(x_+) + \chi_-(x_-)\]

Physical states anti-symmetric in \(b\): \(\chi(b, \phi) = -\chi(b - \pi/2, \phi)\). Imposes relation between \(\chi_+\) and \(\chi_-\)
Wheeler-DeWitt Theory

- Quantum constraint in $b$ representation:
  \[
  \Theta(b)\chi(b, \phi) = -12\pi G \left( b \frac{\partial}{\partial b} b \frac{\partial}{\partial b} \chi(b, \phi) = -\partial^2_{\phi} \chi(b, \phi) \right)
  \]

- As in SLQC, we have an internal clock, physical inner product and Dirac Obsevables.

- Introduce
  \[
  y := (12\pi G)^{-1/2} \ln \left( \frac{b}{2b_0} \right)
  \]
  \[
  \Rightarrow \quad \partial^2_{\phi} \chi(\phi, y) = \partial^2_y \chi(\phi, y)
  \]

- General solution:
  \[
  \chi = \chi_+(\phi + y) + \chi_- (\phi - y) := \chi_+(y_+) + \chi_- (y_-)
  \]

- Unlike SLQC, $\chi_+$ (expanding) and $\chi_-$ (contracting) are disjoint.
**Volume observable in WDW**

\[
(\chi, \hat{V}|\phi \chi)_{\text{phy}} = 2\pi \gamma \ell_p^2 (\hat{v}\chi, \hat{v}\chi)_{\text{kin}}
\]

\[
= \frac{16\gamma \ell_p^2}{\sqrt{12\pi G} b_0} \int_{-\infty}^{\infty} dy_+ \left| \frac{d\chi_+}{dy_+} \right|^2 e^{\sqrt{12\pi G}(\phi-y_+)}
\]

\[
= V_o e^{\sqrt{12\pi G}\phi}.
\]

- As \( \phi \to -\infty, \langle \hat{V}|\phi \rangle \to 0 \). The backward evolution leads to the big bang singularity.

- Fluctuations:

\[
(\chi, \hat{V}^2|\phi \chi)_{\text{phy}} = W_0 e^{2\sqrt{12\pi G}\phi}
\]

\[
\left( (\Delta V|\phi)/\langle \hat{V}|\phi \rangle \right)^2 = (W_0/V_0)^2 - 1.
\]

Remains constant with evolution.
Volume observable in SLQC

\[
(\chi, \hat{V}|\phi \chi)_{\text{phy}} = \frac{8 \gamma \ell_P^2 \lambda^2}{\sqrt{12\pi G}} \left[ \int_{-\infty}^{\infty} dx_+ \left| \frac{d\chi_+}{dx_+} \right|^2 \cosh(\sqrt{12\pi G}(x_+ - \phi)) 
+ \int_{-\infty}^{\infty} dx_- \left| \frac{d\chi_-}{dx_-} \right|^2 \cosh(\sqrt{12\pi G}(-x_- + \phi)) \right]
= I_+ e^{-\sqrt{12\pi G}\phi} + I_- e^{\sqrt{12\pi G}\phi}
\]

There exists a minimum value of \(\langle V|_{(\phi=\phi_B)} \rangle\) which occurs at

\[
\phi_B = (2\sqrt{12\pi G})^{-1} \ln(I_+/I_-)
\]

\(\langle V|_\phi \rangle\) is symmetric across the bounce point.
Fluctuations

\[ \langle V^2 | \phi \rangle = J_0 + J_+ e^{-2\sqrt{12\pi G} \phi} + J_- e^{2\sqrt{12\pi G} \phi} \]

is symmetric across

\[ \phi'_B = (4\sqrt{12\pi G})^{-1} \ln(J_+/J_-) \]

Relative dispersion:

\[ (\Delta V / \langle \hat{V} \rangle)^2_{\phi \to \infty} = \frac{J_-}{I_-^2} - 1 \]

\[ (\Delta V / \langle \hat{V} \rangle)^2_{\phi \to -\infty} = \frac{J_+}{I_+^2} - 1 \]

For \( \phi_B = \phi'_B \), \( D := (\Delta V / \langle \hat{V} \rangle)^2_{\phi \to -\infty} - (\Delta V / \langle \hat{V} \rangle)^2_{\phi \to \infty} = 0 \)

Relative dispersion bounded in time evolution. A single condition on the infinite dimensional space of initial data implies symmetric fluctuations across bounce point.
How much does the Cosmos recall?

For a very large class of states universe retains all its memory across the bounce:

$$\chi(x, \phi) = \int_0^\infty dk \ F(k) e^{-ik(\phi+x)} - \int_0^\infty dk \ \tilde{F}(k) e^{-ik(\phi-x)}$$

For any real and arbitrary $\tilde{F}(k)$, fluctuations are symmetric.
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For any real and arbitrary \(\tilde{F}(k)\), fluctuations are symmetric. Includes real linear combinations of

\[
f_n(k) = k^n e^{-(k-k_0)^2/\beta^2} + ikx_0,
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→ includes all squeezed states with arbitrary squeezing.
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\[ f_n(k) = k^n e^{-(k-k_0)^2/\beta^2+ikx_0}, \]

\[ \rightarrow \text{includes all squeezed states with arbitrary squeezing.} \]

- Cosmos remembers everything across the bounce for such states. There is a Total Recall.
How much does the Cosmos recall?

Consider a general state in the present epoch (post big bang) describing a large classical universe at low curvature

$$\lim_{\phi \to \infty} \left( \frac{\Delta \hat{V}}{\langle \hat{V} \rangle} \right)^2 = \frac{J_0}{I_0^2} - 1 =: \delta_v \ll 1$$

Relative dispersion in curvature:

$$\left( \Delta \tan \left( \frac{\lambda b}{2} \right) / \langle \tan \left( \frac{\lambda b}{2} \rangle \right) \right) = \sqrt{12\pi G} \Delta x =: \delta_b \ll 1$$
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$$\left( \Delta \tan(\lambda_b/2)/\langle \tan(\lambda_b/2) \rangle \right) = \sqrt{12\pi G} \Delta x =: \delta_b \ll 1$$

$$D = \left( \frac{\Delta V}{\langle \hat{V} \rangle} \right)_{\phi \to -\infty} - \left( \frac{\Delta V}{\langle \hat{V} \rangle} \right)_{\phi \to \infty} < (1 + \delta_v) (e^{8\delta_b} - 1)$$
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Difference bounded by the relative dispersions in the initial state. A semi-classical initial state evolves to a semi-classical state after the bounce. Fluctuations are symmetric up to very small difference.
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Difference bounded by the relative dispersions in the initial state. A semi-classical initial state evolves to a semi-classical state after the bounce. Fluctuations are symmetric up to very small difference.

Answer: Universe has a very very sharp memory. Cosmos remembers almost everything after the bounce.
For a fixed value of $\lambda$ select $\Psi_0(b)$: $\langle \hat{V}|_{\phi=0}\rangle_{\lambda} = \langle \hat{V}|_{\phi=0}\rangle_{\text{WDW}} =: V_0$

Relative difference: bounded in future evolution

$$\frac{|\langle \hat{V}\rangle_{\text{WDW}}(\phi) - \langle \hat{V}\rangle_{\lambda}(\phi)|}{\langle \hat{V}\rangle_{\text{WDW}}(\phi)} \leq \delta := I_1/V_0 \text{ (very small)}$$
For a fixed value of \( \lambda \) select \( \Psi_0(b) : \langle \hat{V}_{\phi=0} \rangle_{\lambda} = \langle \hat{V}_{\phi=0} \rangle_{\text{WDW}} =: V_0 \)

Relative difference: bounded in future evolution
\[
|\langle \hat{V}_{\text{WDW}}(\phi) \rangle - \langle \hat{V}_{\lambda}(\phi) \rangle| / \langle \hat{V}_{\text{WDW}}(\phi) \rangle \leq \delta := I_1/V_0 \text{ (very small)}
\]

For a given \( \phi_T \) and \( \epsilon > 0 \), \( \exists \lambda(\epsilon,T) > 0 \) such that,
\[
|\langle \hat{V}_{\text{WDW}}(\phi) \rangle - \langle \hat{V}_{\lambda}(\phi) \rangle| < \epsilon
\]
WDW & SLQC and the lack of continuum limit

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$$|\langle \hat{V}\rangle_{\text{WDW}}(\phi) - \langle \hat{V}\rangle_\lambda(\phi)| < \epsilon$$

For any $N > 0$ (arbitrarily large) $\exists \phi$ such that
$$|\langle \hat{V}\rangle_{\text{WDW}}(\phi) - \langle \hat{V}\rangle_\lambda(\phi)| > N$$
WDW & SLQC and the lack of continuum limit

- For a fixed value of \( \lambda \) select \( \Psi_0(b) : \langle \hat{V} \mid \phi = 0 \rangle_\lambda = \langle \hat{V} \mid \phi = 0 \rangle_{WDW} = : V_0 \)

  Relative difference: bounded in future evolution
  \[ |\langle \hat{V} \rangle_{WDW}(\phi) - \langle \hat{V} \rangle_\lambda(\phi) / \langle \hat{V} \rangle_{WDW}(\phi) | \leq \delta := I_1 / V_0 \text{ (very small)} \]

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  \[ |\langle \hat{V} \rangle_{WDW}(\phi) - \langle \hat{V} \rangle_\lambda(\phi) | > N \]

- Is there a quantum continuum limit of SLQC?

  Consider backward evolution: \( \langle \hat{V} \rangle_{\lambda_0} - \langle \hat{V} \rangle_\lambda \) diverges as \( \phi \to -\infty \).
  \( \phi_B \longrightarrow -\infty \text{ as } \lambda \to 0. \)
  \( \rightarrow \) Uniform limit does not exist.
  Contrast with results on Harmonic Oscillator (Corichi, Vukasinac, Zapata (07)).
Summary

- Bounce not restricted to states which are semi-classical at late times. There is a pre-big bang branch for a dense subspace of $\mathcal{H}_{\text{phy}}$.
- For a very large class of states fluctuations are exactly symmetric across the bounce. More general states which describe a large volume low curvature epoch, fluctuations are symmetric up to negligible difference. Universe retains almost all its memory across bounce. (Results in harmony with various numerical simulations).
- SLQC and WDW approach GR at low curvatures. At large curvatures they depart significantly.
- In the backward evolution of the expanding branch for any given fixed time interval, SLQC and WDW agree to arbitrary accuracy by a choice of $\lambda$. However, for any given choice of $\lambda$, they diverge if one waits long enough.
- There is no limiting theory of SLQC when $\lambda \to 0$. Two different $\lambda$ SLQC’s depart in a similar way as they do from WDW. SLQC is a fundamentally discrete theory.
Start with an arbitrary $\lambda_o$, refine the area gap ($\lambda_o \rightarrow \lambda$). For $\lambda < \lambda_o$, $\chi_i \in \mathcal{H}_{\lambda_o}$ under embedding $\chi_i \in \mathcal{H}_{\lambda}$. Under renormalization $\chi^\lambda := \sqrt{\lambda_o/\lambda} \chi^{\lambda_o}$, $|\chi^\lambda|^2 = |\chi^{\lambda_o}|^2$. 
Fundamental discreteness of SLQC

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- Initial data: At $\phi = \phi_i$, $\chi^\lambda_i$ and $\chi^{\lambda_o}_i$ give same $\langle \hat{V} \rangle$. 
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- Refinement in $\lambda \Rightarrow x_\lambda \neq x_{\lambda_0} \Rightarrow I_{+, -}(\lambda) \neq I_{+, -}(\lambda_0)$. $I_{-}(\lambda)$ is a monotonic decreasing function. As $\lambda \rightarrow 0$, $I_{-}(\lambda)$ grows.
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Refinement in $\lambda \Rightarrow x_\lambda \neq x_{\lambda_o} \Rightarrow I_{+,-}(\lambda) \neq I_{+,-}(\lambda_o)$. $I_{-}(\lambda)$ is a monotonic decreasing function. As $\lambda \rightarrow 0$, $I_{-}(\lambda)$ grows.

Consequence: In the backward evolution of an expanding branch

$$\langle \hat{V} \rangle_{\lambda_o} - \langle \hat{V} \rangle_{\lambda} = (I_{-}(\lambda_o) - I_{-}(\lambda))e^{-\sqrt{12\pi G} \phi}$$

which diverges as $\phi \rightarrow -\infty$.

$$\phi_B = (2\sqrt{12\pi G})^{-1} \ln(I_{+}/I_{-}) \rightarrow -\infty \quad \text{as} \quad \lambda \rightarrow 0.$$
Fundamental discreteness of SLQC

- Start with an arbitrary \( \lambda_o \), refine the area gap (\( \lambda_o \to \lambda \)).
  For \( \lambda < \lambda_o \), \( \chi_i \in \mathcal{H}_{\lambda_o} \) under embedding \( \chi_i \in \mathcal{H}_\lambda \).
  Under renormalization \( \chi^\lambda := \sqrt{\lambda_o/\lambda} \chi^{\lambda_o} \), \( |\chi^\lambda|^2 = |\chi^{\lambda_o}|^2 \).

- Initial data: At \( \phi = \phi_i \), \( \chi_i^\lambda \) and \( \chi_i^{\lambda_o} \) give same \( \langle \hat{V} \rangle \).

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  \phi_B = (2\sqrt{12\pi G})^{-1} \ln(I_+/I_-) \to -\infty \quad \text{as} \quad \lambda \to 0.
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- Uniform limit does not exist. Contrast with results on Harmonic Oscillator (Corichi, Vukasinac, Zapata (07)).